M. Math IInd Yr Back paper Exam 2016 Advanced Functional Analysis

Answer all the questions. Each question is worth 10 points. You may state correctly and use any result proved in the class. However if an answer is an almost immediate consequence of the stated result, such a result also need to be proved. *All topological spaces are assumed to be Hausdorff.*

1) Let X be a complex normed linear space, Y a Banach space. Suppose the space of operators $\mathcal{L}(X, Y)$ is equipped with the usual norm. Show that it is a Banach space.

2) Let X be a locally convex topological vector space. Let $A, B \subset X$ be compact convex sets. Show that the convex hull $CO(A \cup B)$ is a closed set.

3) Let X be a locally convex topological vector space. Let $F \subset X$ be a finite dimensional subspace. Show that there is a closed subspace $Y \subset X$ such that $X = F \bigoplus Y$.

4) Let X, Y be Banach spaces. Let $T : X \to Y$ be a surjective isometry. Suppose X^* and Y^* are equipped with the weak*-topology. Show that $T^* : Y^* \to X^*$ is an isomorphism.

5) Give an example of a Banach space X such that X is not onto isometric to X^*.

6) Let Ω be an infinite compact set. Show that $C(\Omega)$ is an infinite dimensional space.

7) Show that for any closed and bounded set $A \subset R^n$, $CO(A)$ is a closed set.

8) Let X, Y be Banach spaces. Show that if $T \in \mathcal{L}(X, Y)$ has closed range, then $T^*(Y^*)$ is a weak*-closed subspace of X^*.

9) Let $T : X \to X$ be a compact operator and $\lambda \neq 0$. Show that $T - \lambda I$ has closed range.

10) Let $T \in \mathcal{L}(X)$ be such that T^* maps extreme points of the unit ball of X^* to extreme points of the unit ball of X^*. Show that T is an extreme point of the unit ball of $\mathcal{L}(X)$.